CSP reductions

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Graphs and homomorphisms

Definition

- A **directed graph** is a pair $\mathbb{G} = (G; E)$, where G is the set of vertices and $E \subseteq G^2$ is the set of edges.
- A **relational structure** is a tuple $\mathbb{G} = (G; E_1, \dots, E_k)$, where G is the underlying set and $E_i \subseteq G^{n_i}$ is an n_i-ary relation.

Definition

A **homomorphism** from $\mathbb{G} = (G; E)$ to $\mathbb{H} = (H; F)$ is a map $f : G \to H$ that preserves edges

$$(a, b) \in E \quad \Longrightarrow \quad (f(a), f(b)) \in F.$$

We write $\mathbb{G} \to \mathbb{H}$ if there exists a homomorphism from $\mathbb{G}$ to $\mathbb{H}$.

Constraint satisfaction problem (CSP)

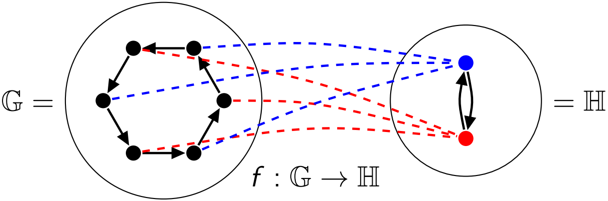
Definition

For a finite relational structure $\mathbb{H}$ we define

$$\mathrm{CSP}(\mathbb{H}) = \{\, \mathbb{G} \mid \mathbb{G} \to \mathbb{H} \,\}.$$

Example

- CSP() is the class of three-colorable graphs.
- CSP() is the class of bipartite graphs.



The computational complexity of CSP

The membership problem for $\mathrm{CSP}(\mathbb{H})$

- always decidable in nondeterministic polynomial time (**NP**)
- is decidable in polynomial time (**P**) for some $\mathbb{H}$

Dichotomy Conjecture (T. Feder, M. Vardi, 1993)

For every finite structure $\mathbb{H}$ the membership problem for $\mathrm{CSP}(\mathbb{H})$ is in **P** or **NP**-complete.

The dichotomy conjecture holds when $\mathbb{H}$

- is an undirected graph (P. Hell, J. Nešetřil, 1990), or
- has at most 3 elements (A. Bulatov, 2006), or
- a smooth directed graph (L. Barto, M. Kozik, T. Niven, 2009).

Open for directed graphs.

Example: solving a system of equations

$$(\exists x, y, z \in \mathbf{Z}_5)(x + y = z \ \wedge \ x + x = y \ \wedge \ z = 1)$$

$$\Updownarrow$$

$$(\exists x, y, z \in \mathbf{Z}_5)((x, y, z) \in F_1 \ \wedge \ (x, x, y) \in F_1 \ \wedge \ z \in F_2),$$

where $F_1 = \{ (x, y, z) \in \mathbf{Z}_5^3 : x + y = z \}$ and $F_2 = \{1\}$.

$$\Updownarrow$$

$$(\exists f : \{1, 2, 3\} \to \mathbf{Z}_5)((f(1), f(2), f(3)) \in F_1 \ \wedge$$
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where $\mathbb{G} = (\{1, 2, 3\}; E_1, E_2)$, $\mathbb{H} = (\mathbf{Z}_5; F_1, F_2)$

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$$\mathbb{G} \in \mathrm{CSP}(\mathbb{H})$$

CSP REDUCTIONS: CORES

Lemma

For every finite relational structure $\mathbb{H}_1$ *there exists* $\mathbb{H}_2$ *such that*

1. $\mathbb{H}_2$ *is a directed graph (with unary relations),*
2. $\mathbb{H}_2$ *is a* **core**, *i.e., every endomorphism is bijective,*
3. *every singleton unary relation* $\{a\}$ *is in* $\mathbb{H}_2$, *and*

$\mathrm{CSP}(\mathbb{H}_1)$ *is polynomial time equivalent to* $\mathrm{CSP}(\mathbb{H}_2)$.

Proof of (2).

Take a homomorphism $\mathbb{H}_1 \to \mathbb{H}_2$ where $\mathbb{H}_2$ is a substructure of $\mathbb{H}_1$ of minimal size. Then $\mathbb{H}_2$ is a core by minimality. The natural embedding $\mathbb{H}_2 \to \mathbb{H}_1$ is also a homomorphism. Therefore $\mathbb{H}_1 \leftrightarrow \mathbb{H}_2$ and consequently $\mathrm{CSP}(\mathbb{H}_1) = \mathrm{CSP}(\mathbb{H}_2)$. □

CSP reductions: polymorphisms

Definition

A **polymorphism** of $\mathbb{H} = (H; F)$ is a homomorphism $p : \mathbb{H}^n \to \mathbb{H}$, that is a n-ary map that preserves edges

$$(a_1, b_1), \ldots, (a_n, b_n) \in F \implies (p(a_1, \ldots, a_n), p(b_1, \ldots, b_n)) \in F.$$

$\mathrm{Pol}(\mathbb{H}) = \{ p \mid p : \mathbb{H}^n \to \mathbb{H} \}$ is the **clone of polymorphisms**.

Lemma

If $\mathrm{Pol}(\mathbb{H}_1) \subseteq \mathrm{Pol}(\mathbb{H}_2)$, *then* $\mathrm{CSP}(\mathbb{H}_2)$ *is polynomial time reducible to* $\mathrm{CSP}(\mathbb{H}_1)$.

Question

Which polymorphisms guarantee that $\mathrm{CSP}(\mathbb{H})$ is in **P**?

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Nice polymorphisms

Theorem

$\mathrm{CSP}(\mathbb{H})$ is in **P** *if* $\mathrm{Pol}(\mathbb{H})$ *contains one of the following:*

- *a semilattice operation* (Jevons et. al.)

$$x \wedge y \approx y \wedge x, \quad x \wedge (y \wedge z) \approx (x \wedge y) \wedge z, \quad x \wedge x \approx x.$$

- *a near-unanimity operation*

$$p(y, x, \ldots, x) \approx p(x, y, x, \ldots, x) \approx \cdots \approx p(x, \ldots, x, y) \approx x,$$

- *a totally symmetric idempotent operation* (Dalmau, Pearson),
- *a Maltsev operation* (Bulatov, Dalmau)

$$p(x, y, y) \approx p(y, y, x) \approx x,$$

- *Generalized majority-minority operation* (Dalmau).

Nice polymorphisms cont.

Theorem

$\mathrm{CSP}(\mathbb{H})$ *is in* **P** *if* $\mathrm{Pol}(\mathbb{H})$ *contains one of the following:*

- *Edge operations* (Idziak, Marković, McKenzie, Valeriote, Willard)

$$
\begin{aligned}
p(y, y, x, x, \dots, x) &\approx x, \\
p(x, y, y, x, \dots, x) &\approx x, \\
p(x, x, x, y, \dots, x) &\approx x, \\
&\vdots \\
p(x, x, x, x, \dots, y) &\approx x.
\end{aligned}
$$

- *Jónsson operations* (Barto, Kozik),
- *Willard operations* (Barto, Kozik).

Weak near-unanimity

Theorem (R. McKenzie, M. Maróti, 2008)

For a locally finite variety $\mathcal{V}$ the followings are equivalent:

- *$\mathcal{V}$ omits type **1** (tame congruence theory),*
- *$\mathcal{V}$ has a Taylor term,*
- *$\mathcal{V}$ has a **weak near-unanimity** operation:*

$$p(y, x, \ldots, x) \approx \cdots \approx p(x, \ldots, x, y) \quad \text{and} \quad p(x, \ldots, x) \approx x.$$

Theorem (B. Larose, L. Zádori, 2006)

*If $\mathbb{H}$ is a core and does not have a Taylor (or weak near-unanimity) polymorphism, then $\mathrm{CSP}(\mathbb{H})$ is **NP**-complete.*

Algebraic dichotomy conjecture

If $\mathbb{H}$ is a core and has a weak near-unanimity polymorphism, then $\mathrm{CSP}(\mathbb{H})$ is in **P**.

Applications to universal algebra

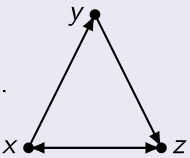
Theorem (P. Markovic, R. McKenize, M. Siggers, 2008)

A locally finite variety $\mathcal{V}$ omits type **1** *iff it has a 4-ary term t satisfying the equations*

$$t(x,y,z,x) \approx t(y,z,x,z) \quad \textit{and} \quad t(x,x,x,x) \approx x.$$

Proof.

Consider the directed graph $\mathbb{G}$ defined on the 3-generated free algebra $\mathbf{F}_3(\mathcal{V})$ whose edges are generated by (x,y), (y,z), (z,x), (x,z). It is smooth, and its core must be a loop. That loop edge is $t((x,y),(y,z),(z,x),(x,z))$.



□

Strategies

Definition

Let $\mathbf{H}$ be an idempotent algebra and G be a set of variables. A collection

$$\mathcal{R} = \{\, \mathbf{R}_{ij} \leq \mathbf{H}^2 : i, j \in G \,\}$$

of binary constraint relations is a **binary strategy**, if

- $R_{ij} = R_{ji}^{-1}$, and
- $R_{ii} \subseteq \{\, (a, a) : a \in H \,\}$.

A map $f : G \to H$ is a **solution** if $(f(i), f(j)) \in R_{ij}$ for all $i, j \in G$.

Lemma

Every CSP *problem is polynomial time equivalent to*

$$\mathrm{CSP}(\mathrm{S}(\mathbf{H}^2)) = \{\textit{all binary strategies of } \mathbf{H} \textit{ that have a solution}\}.$$

Local consistency algorithm

Definition

The binary strategy $\mathcal{R}$ is **(2,3)-consistent**, if

$$R_{ik} \subseteq R_{ij} \circ R_{jk} \quad \text{for all } i, j, k \in G.$$

Definition

The **(2,3)-consistency algorithm** turns a binary strategy $\mathcal{R}$ into a (2,3)-consistent binary strategy without loosing solutions:

$$R'_{ik} = R_{ik} \cap (R_{ij} \circ R_{jk}).$$

- runs in polynomial time (in the size of $\mathcal{R}$),
- the output is independent of the choices made,
- if the output strategy is empty, then $\mathcal{R} \notin \mathrm{CSP}(\mathrm{S}(\mathbf{H}^2))$.

Bounded width

Definition

An algebra **H** has **width** $(2,3)$ if every nonempty $(2,3)$-consistent binary strategy of **H** has a solution.

Theorem (A. Bulatov, A. Krokhin, P. Jeavons, 2000)

If **H** *has width* $(2,3)$ *(or bounded width), then*

- $\mathrm{CSP}(\mathrm{S}(\mathbf{H}^2))$ *is in* **P**
- $\mathrm{HSP}(\mathbf{H})$ *omits types* **1** *and* **2**, *i.e.,* **H** *has Willard terms.*

Theorem (L. Barto, M. Kozik, 2009)

If the variety generated by **H** *omits types* **1** *and* **2**, *then* **H** *has bounded width.*

Consistent maps

Definition

Let $\mathcal{R} = \{ \mathbf{R}_{ij} \leq \mathbf{H}^2 \mid i, j \in G \}$ be a binary strategy. A collection of maps

$$\mathcal{P} = \{ p_i : H \to H \mid i \in G \}$$

is **consistent**, if $p_i \times p_j$ preserves $\mathbf{R}_{ij}$ for all $i, j \in G$, i.e.

$$(a, b) \in \mathbf{R}_{ij} \implies (p_i(a), p_j(b)) \in \mathbf{R}_{ij}.$$

- The identity maps $p_i(x) = x$ are always consistent.
- If f is a solution of $\mathcal{R}$, then the constant maps $p_i(x) = f(i)$ are consistent.
- Consistent maps can be composed pointwise.
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Finding consistent maps

Consistent maps are solutions of a larger CSP instance:

Definition

Let $\mathcal{R} = \{\, \mathbf{R}_{ij} \le \mathbf{H}^2 \mid i, j \in G \,\}$ be a binary strategy. A collection of maps $\{\, p_i \mid i \in G \,\}$ is consistent if and only if the binary strategy $\mathcal{R}'$ with

- variable set $G' = G \times H$ and
- relations

$$\mathbf{R}'_{(ia)(jb)} = \begin{cases} \mathbf{R}_{ij} & \text{if } (a, b) \in \mathbf{R}_{ij}, \\ \mathbf{H} \times \mathbf{H} & \text{otherwise} \end{cases}$$

has

$$f'((i, a)) = p_i(a)$$

as a solution.

Finding consistent maps cont.

Lemma

For any binary term $t(x, y)$ and a solution f of the binary strategy $\mathcal{R} = \{ \mathbf{R}_{ij} \leq \mathbf{H}^2 \mid i, j \in G \}$ the maps

$$\{ p_i(x) = t(f(i), x) \mid i \in G \}$$

are consistent.

- We can assume that $t(x, t(x, y)) = t(x, y)$ in which case the consistent maps become idempotent $p_i(p_i(y)) = p_i(y)$.
- We can add non-unary constraints to $\mathcal{R}'$ such that p_i become polynomials.
- We can limit the domain of the variable $(i, a) \in G'$ to

$$t(a, H) = \{ t(a, y) \mid y \in H \}.$$

Using consistent maps

- We have to use multisorted strategies.
- We decompose CSP problems into easier problems.
- We assume that "smaller" strategies are in **P**.
- If $|t(a, H)| < |H|$ for all $a \in H$, then we are in a "smaller" case (e.g. algebras with a non-chain semilattice factor)
- So we can find a consistent set $\mathcal{P}$ of maps for $\mathcal{R}$.
- $\mathcal{P}$ maps solutions to solutions, so we can find a solution of $\mathcal{R}$ in a smaller case provided that $|\{ p_i(x) : x \in H \}| < |H|$.
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Results

Theorem

Let $\mathbf{H}$ *be an algebra and* t *be a binary term such that for each* $a \in H$ *the map* $t_a(x) = t(a, x)$ *is idempotent and not surjective. Let* B *be the set of elements* $b \in H$ *such that the map* $x \mapsto t(x, b)$ *is a permutation. If* B *generates a proper subuniverse of* $\mathbf{H}$*, then* $\mathbf{H}$ *can be eliminated from* CSP *problems.*

Applications:

- three element structures,
- tree over Maltsev,
- conservative algebras (partially).